Generalized Representation Theorem and its Applications

Abstract

In this paper we will discuss some generalizations of the division with remainder. Namely, we will look at the **Generalized Representation Theorem** and at the **Weighted Generalized Representation**. We will finish off with some applications to solving of mathematical Olympiad style problems related the the wellknown Coin Problem, also known as the Diophantine Frobenius Problem (FP).

1 Introduction

In the initial stages of learning mathematics, division with remainder is perceived as an empirical fact, and pretty soon becomes a familiar technical detail. In reality it is a theorem (the **Representation Theorem**) which reflects the fundamental properties of integers. It underlies not only the theory of numbers, giving rise to the fundamental theorems and constructions, such as Euclid's algorithm and the positional number system, but also mathematics in general.

The **Generalized Representation Theorem,** considered below owes its existence to the **Representation Theorem.**

2 Definitions, Facts, Notations, and Agreements

2.1 Representation Theorem (RT).

Theorem 1. *For any two integers a and b, where $b \neq 0$, there is a unique pair (k, r) of integers, such that,*

$$\begin{cases} a = kb + r \\ 0 \leq r < |b| \end{cases} \iff \begin{cases} \dfrac{a}{b} = k + \dfrac{r}{b} \\ 0 \leq r < |b| \end{cases}.$$

Here a and b are the *dividend* and *divisor*, respectively. The numbers $k = k_b(a)$ and $r = r_b(a)$ we call the *quotient* and *remainder*, respectively. If the remainder $r_b(a) = 0$ we say that b divides a or a is divisible by b with notation $b \mid a$ or $a \,\vdots\, b$.

Let $D(a)$ be the set of positive divisors of the integer a. For any subset $\mathcal{A} \subset \mathbb{Z}$ let $D(\mathcal{A})$ be the set of positive divisors which are common for all $a \in \mathcal{A}$, that is,

$$D(\mathcal{A}) = \bigcap_{a \in \mathcal{A}} D(a).$$

We also let

$$d(\mathcal{A}) = \max D(\mathcal{A}) = \gcd(\mathcal{A})$$

be the greatest common positive divisor of $\mathcal{A}$. In particular, if $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ we write $D(a_1, a_2, \ldots, a_n)$ instead of $D(\mathcal{A})$ and $d(a_1, a_2, \ldots, a_n)$ instead of $d(\mathcal{A})$. When $\mathcal{A} = \{a\}$ we define $d(a) = a$. Since (follows from Euclid's algorithm)

$$D(a, b) = D(d(a, b)),$$

by math induction it is not difficult to prove that

$$D(\mathcal{A}) = D(d(\mathcal{A})).$$

From the definition it also follow that

$$D(\mathcal{A} \cup \mathcal{B}) = D(\mathcal{A}) \cap D(\mathcal{B}).$$

We say that set of integers $\mathcal{A}$ is *coprime* if $d(\mathcal{A}) = 1$ and as *totally coprime* if for any two distinct numbers a and b from $\mathcal{A}$, we have $\gcd(a, b) = 1$(alternative usage is $d(a, b) = 1$ or the notation $a \perp b$).

2.2 Properties of $d(a,b)$

1. For any integer $k,\ d(a,b) = d(a-kb,b)$. This is also know as the *Preservation Property.*
2. There are integers x and y such that $d(a,b) = ax+by$. This is also known as the *Linear Representation* of $d(a,b)$. In particular, if $d(a,b)=1$ then $ax+by=1$, and if for some integers $x,y,ax+by=1$ then $d(a,b)=1$.
3. $d(ka,kb) = kd(a,b)$ for any $k \in \mathbb{N}$.
4. $d(a,b)=1$ and $d(a,c)=1$ if and only if $d(a,bc)=1$.
5. If $d(ab,c)=1$ and $d(b,c)=1$ then $d(a,c)=1$.
6. If $d(b,c)=1$ and b,c divide a, then bc divides a.
7. If bc divides a and $d(b,a)=1$, then c divides a.
8. For any two sets of integers $\mathcal{A},\mathcal{B}$
$$d(d(\mathcal{A}), d(\mathcal{B})) = d(\mathcal{A}\cup\mathcal{B}).$$
Indeed, since
$$D(d(\mathcal{A}), d(\mathcal{B})) = D(d(\mathcal{A})) \cap D(d(\mathcal{B})) = D(\mathcal{A}) \cap D(\mathcal{B}) = D(\mathcal{A}\cup\mathcal{B}),$$
then
$$d(d(\mathcal{A}), d\mathcal{B})) = \max\{D(d(\mathcal{A}), d(\mathcal{B})\} = \max\{D(\mathcal{A}\cup\mathcal{B})\} = d(\mathcal{A}\cup\mathcal{B}).$$

3 Generalized Representation Theorem (GRT).

Theorem 2. *Let m be an integer and let $\{a_1, a_2, \ldots, a_n\}$ be a totally coprime set of positive integers. Then there are unique integers $k, r_1, r_2, \ldots, r_n$ such that*
$$\frac{m}{a_1 a_2 \cdots a_n} = k + \frac{r_1}{a_1} + \frac{r_2}{a_2} + \cdots + \frac{r_n}{a_n},$$
where $0 \le r_i < a_i, i = 1,2,\ldots,n$.

Proof. We will prove existance by mathematical induction.

The Base Case. Let $n=2$. Since $d(a_1,a_2)=1$, there are integers x_1,x_2 such that $x_1a_1+x_2a_2=1$. By **RT**, we have
$$mx_1 = k_1a_2 + r_1$$
$$mx_2 = k_2a_1 + r_2,$$
with $0 \le r_i < a_i,\ i=1,2$. Then
$$\begin{aligned} m &= mx_1a_1 + mx_2a_2 = a_1(k_2a_2+r_2) + a_2(k_1a_1+r_1) \\ &= a_1a_2(k_1+k_2) + a_1r_2 + a_2r_1. \end{aligned}$$
Hence,
$$\frac{m}{a_1a_2} = k + \frac{r_1}{a_1} + \frac{r_2}{a_2},$$
where $k = k_1 + k_2$.

The Step Case. Let $\{a_1, a_2, \ldots, a_n, a_{n+1}\}$ be a totally coprime set of $n+1$ positive integers. By the assumption of the induction there are integers $k, r_1, r_2, \ldots, r_n$ such that

$$\frac{m}{a_1 a_2 \cdots a_n} = k + \sum_{k=1}^{n} \frac{r_i}{a_i},$$

where $0 \le r_i < a_i, i = 1, 2, \ldots, n$. Then

$$\frac{m}{a_1 a_2 \cdots a_n a_{n+1}} = \frac{k}{a_{n+1}} + \sum_{k=1}^{n} \frac{r_i}{a_i a_{n+1}}.$$

By **RT**, there are integers l and r_{n+1} such that

$$\frac{k}{a_{n+1}} = l + \frac{r_{n+1}}{a_{n+1}},$$

where $0 \le r_{n+1} < a_{n+1}$. By the base case there are integers l_i, t_i, s_i such that $0 \le t_i < a_i, 0 \le s_i < a_{n+1}$ and

$$\frac{r_i}{a_{n+1} a_i} = l_i + \frac{t_i}{a_i} + \frac{s_i}{a_{n+1}}, i = 1, 2, \ldots, n.$$

By **RT**, there are l_{n+1}, t_{n+1} such that

$$\frac{r_{n+1} + \sum_{i=1}^{n} s_i}{a_{n+1}} = l_{n+1} + \frac{t_{n+1}}{a_{n+1}},$$

where $0 \le t_{n+1} < a_{n+1}$. Finally,

$$\begin{aligned} \frac{m}{a_1 a_2 \cdots a_n a_{n+1}} &= \left(l + \frac{r_{n+1}}{a_{n+1}}\right) + \left(\sum_{i=1}^{n} l_i + \sum_{i=1}^{n} \frac{t_i}{a_i} + \sum_{i=1}^{n} \frac{s_i}{a_{n+1}}\right) \\ &= l + \sum_{i=1}^{n} l_i + \sum_{i=1}^{n} \frac{t_i}{a_i} + \frac{\sum_{i=1}^{n} s_i + r_{n+1}}{a_{n+1}} = l + \sum_{i=1}^{n+1} l_i + \sum_{i=1}^{n+1} \frac{t_i}{a_i}. \end{aligned}$$

where $0 \le t_i < a_i$ and $i = 1, 2, \ldots, n+1$.

Let us now prove uniqueness. Suppose that there are two representations of

$$\frac{m}{a_1 a_2 \cdots a_n} = k + \sum_{i=1}^{n} \frac{r_i}{a_i} = h + \sum_{i=1}^{n} \frac{s_i}{a_i},$$

where $0 \le r_i, s_i < a_i, i = 1, 2, \ldots, n$. Let $A_i = \dfrac{a_1 a_2 \cdots a_n}{a_i}, i = 1, 2, \ldots, n$. Since $k - h = \sum_{i=1}^{n} \dfrac{s_i - r_i}{a_i}$ and $n_{ij} = \dfrac{A_j}{a_i} \in \mathbb{Z}$, for $i \ne j$, then

$$A_j(k-h) = \sum_{i=1}^{n} \frac{A_j(s_i - r_i)}{a_i} \iff A_j(k-h) = \sum_{i=1, i\ne j}^{n} \left(n_{ij}(s_i - r_i) + \frac{A_j(s_j - r_j)}{a_j}\right).$$

Because $\dfrac{A_j(s_j - r_j)}{a_j}$ is an integer and $d(A_j, a_j) = 1$, we have that a_j divides $|s_j - r_j| < a_j,\ j = 1, 2, \ldots, n$, which is a contradiction.

Corollary 3 (Weighted Generalized Representation (WGR).)**.** *Let $\{a_1, a_2, \ldots, a_n\}$ be a totally coprime set of positive integers and let $p_1, p_2, \ldots, p_n$ be positive integers such that $p_i \perp a_i, i = 1, 2, \ldots, n$. Then there are unique integers $l, t_1, t_2, \ldots, t_n$ such that*

$$\frac{m}{a_1 a_2 \cdots a_n} = l + \frac{p_1 t_1}{a_1} + \frac{p_2 t_2}{a_2} + \cdots + \frac{p_n t_n}{a_n},$$

where $0 \le t_i < a_i, i = 1, 2, \ldots, n$.

Proof. By **GRT**, there are $k, r_1, r_2, \ldots, r_n$ such that

$$\frac{m}{a_1 a_2 \cdots a_n} = k + \sum_{i=1}^{n} \frac{r_i}{a_i},$$

where $0 \le r_i < a_i,\ i = 1, 2, \ldots, n$. For each $i = 1, 2, \ldots, n$ there are l_i, s_i, t_i such that $0 \le s_i < p_i$, $0 \le t_i < a_i$ and

$$\frac{r_i}{a_i p_i} = l_i + \frac{s_i}{p_i} + \frac{t_i}{a_i} \iff \frac{r_i}{a_i} = l_i p_i + s_i + \frac{p_i t_i}{a_i}.$$

Then

$$\frac{m}{a_1 a_2 \cdots a_n} = l + \sum_{i=1}^{n} \frac{p_i t_i}{a_i},$$

where $l = k + \sum_{i=1}^{n} (l_i p_i + s_i)$.

Uniqueness can be proven similarly as the uniqueness part of **GRT.**

One application of the **WGR** is the **Chinese Remainder Theorem (CRT)**. Indeed, let $x \equiv r_i \pmod{a_i}$, $i = 1, 2, \ldots, n$, and let $\{a_1, a_2, \ldots, a_n\}$ be a totally coprime set. Let $A = a_1 a_2 \cdots a_n$ and let $A_i = \frac{A}{a_i}$. Then by the **WGR** with weights p_i such that $p_i A_i \equiv 1 \pmod{a_i}$, $i = 1, 2, \ldots, n$ we can write x as

$$\frac{x}{A} = l + \sum_{i=1}^{n} \frac{p_i t_i}{a_i} \Leftrightarrow x = Al + A \sum_{i=1}^{n} \left(\frac{p_i t_i}{a_i} \right) = l a_1 a_2 \cdots a_n + \sum_{i=1}^{n} A_i p_i t_i$$

for some integers $l, t_1, t_2, \ldots, t_n$, where $0 \le t_i < a_i$ for $i = 1, 2, \ldots, n$. For $i = 1, 2, \ldots, n$, from this representation, it follows that

$$x \equiv t_i \pmod{a_i}$$

and from the condition $x \equiv r_i \pmod{a_i}$. Then

$$r_i \equiv t_i \pmod{a_i},$$

where $0 \le t_i, r_i < a_i$, and we conclude that that $r_i = t_i$. Thus

$$x = l a_1 a_2 \cdots a_n + \sum_{i=1}^{n} A_i p_i r_i,$$

which is the Lagrange form of the Chinese Remainder Theorem.

4 Applications.

The following applications are Olympiad style problems related to the Diophantine Frobenius Problem (FP).

4.1 Definitions

Let $\mathcal{A}$ =$\{a_1, a_2, \ldots, a_n\}$ be a coprime set of positive integers and let $G(\mathcal{A})$ be the set of all linear combinations of $a_1, a_2, \ldots, a_n$ with nonnegative integer coefficients, that is

$$G(\mathcal{A}) = \{a_1x_1 + a_2x_2 + \cdots + a_nx_n \mid x_1, x_2, \ldots, x_n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}.$$

A number which belongs to $G(\mathcal{A})$ will be called an $\mathcal{A}-$*proper* number and all other integers will be called $\mathcal{A}-$*improper* numbers or, simply proper and improper, respectively.

It is not difficult to see that any negative integers are improper and the smallest proper number is nonnegative. Since $G(\mathcal{A})$ is an almost inductive set, i.e., there is an integer m such that

$$\mathbb{Z}_{>m} = \{m+1, m+2, m+3, \ldots\} \in G(\mathcal{A})$$

then, by the well ordering principle, we define smallest integer number μ with property

$$\mathbb{Z}_{>\mu} \in G(\mathcal{A})$$

which is at the same time the greatest $\mathcal{A}-$ improper number. That is

$$\mu = \mu(\mathcal{A}) = \max \mathbb{Z} \diagdown G(\mathcal{A}).$$

Then $\mu(\mathcal{A}) = -1$ if $1 \in \mathcal{A}$.

The general *Frobenius Problem* is:

Find the explicit form of $\mu(\mathcal{A})$ *as function of the coprime set* $\mathcal{A}$ =$\{a_1, a_2, \ldots, a_n\}$.

The general case of this problem is still open and even for $n = 3$ there are many algorithms but not explicit formula. But under special conditions on $a_1, a_2, \ldots, a_n$, the **FP** can be solved successfully. Following are some of these special cases along with solutions.

4.2 Problems

Problem 1 For any two coprime positive integers a and b find the explicit form of

$$\mu(a, b).$$

Solution. Let $a, b \in \mathbb{N}$ and $a \perp b$. Consider the equation $ax + by = m$ in $\mathbb{N}_0$, where $m \in \mathbb{N}_0$. By the **RT** and the **GRT** there are integers numbers $t \geq 0, s \geq 0, k, p, p', q, q'$ such that $x = bt + q', y = as + p'$ and $\dfrac{m}{ab} = k + \dfrac{p}{a} + \dfrac{q}{b}$, with $0 \leq p', p < a, 0 \leq q', q < b$. Then

$$\begin{aligned} ax + by = m &\iff t + s + \frac{p'}{a} + \frac{q'}{b} = k + \frac{p}{a} + \frac{q}{b} \\ &\iff \begin{cases} t + s = k \\ p = p', q = q' \end{cases}. \end{aligned}$$

The last system of equations is true from the uniqueness of representations. Thus the equation $ax+by = m$ is solvable in $\mathbb{N}_0$ if and only if the equation

$$t + s = k \tag{1}$$

it is solvable in $\mathbb{N}_0$. Since the -1 greatest integer k for which the equation is unsolvable in $\mathbb{N}_0$, then greatest value of m for which equation the $ax + by = m$ is unsolvable in $\mathbb{N}_0$ is

$$ab\left(-1 + \frac{a-1}{a} + \frac{b-1}{b}\right) = ab - a - b.$$

Thus

$$\mu(a,b) = ab - a - b$$

and

$$\mathbb{Z}_{>\mu(a,b)} \subset G(a,b).$$

This formula was discovered by J. J. Sylvester in 1884.

From (1) it follows that the equation $ax + by = m$ has exactly k solutions in $\mathbb{N}_0$ if and only if $m = (k-1)ab + aq + bp$, where p, q are any integers satisfying $0 \le p < a, 0 \le q < b$.

Problem 2 Find the number $N(a,b)$ of nonnegative (a,b)-improper numbers, i.e., values of nonnegative m for which the equation $ax + by = m$ is unsolvable in $\mathbb{N}_0$.

Solution. Since $ax + by = m$ is unsolvable in $\mathbb{N}_0$ if and only if

$$m = -ab + qa + pb \ge 0 \iff qa + pb \ge ab$$

and $0 \le p \le a-1, 0 \le q \le b-1$, then by letting $t = a - p$, we obtain

$$m = qa - bt,$$

where $0 \le q \le b-1, 1 \le t \le a$ and $qa - bt \ge 0 \iff t \le \left\lfloor \frac{aq}{b} \right\rfloor$. Since $\frac{aq}{b} \le \frac{a(b-1)}{b} < a$ then $m = qa - bt$, where $0 \le q \le b-1$ and $1 \le t \le \left\lfloor \frac{aq}{b} \right\rfloor$. Therefore,

$$\begin{aligned} N(a,b) &= \sum_{q=0}^{b-1} \sum_{t=1}^{\lfloor aq/b \rfloor} 1 = \sum_{q=0}^{b-1} \left\lfloor \frac{aq}{b} \right\rfloor = \sum_{q=1}^{b-1} \left\lfloor \frac{aq}{b} \right\rfloor = \sum_{q=1}^{b-1} \left(\frac{aq}{b} - \left\{ \frac{aq}{b} \right\} \right) \\ &= \frac{a}{b} \cdot \frac{b(b-1)}{2} - \sum_{q=1}^{b-1} \left\{ \frac{aq}{b} \right\} = \frac{a(b-1)}{2} - \sum_{q=1}^{b-1} \frac{r_b(aq)}{b} = \frac{a(b-1)}{2} - \sum_{q=1}^{b-1} \frac{q}{b} \\ &= \frac{a(b-1)}{2} - \frac{b-1}{2} = \frac{(a-1)(b-1)}{2}, \end{aligned}$$

because $\{r_b(aq) \mid 1 \le q \le b-1\} = \{1, 2, \ldots, b-1\}$.

Problem 3 For given positive integer numbers k and $a \perp b$ find the greatest integer m for which the system (2)

$$\begin{cases} ay + bx = m \\ 0 \le x \le ky \end{cases}$$

is unsolvable in $\mathbb{Z}$.

Solution. As in the solution to Problem 1, by the **RT** and by the **GRT** we have uniquely determined integers $t \ge 0, s \ge 0, l, p, q$, such that $x = bt + q, y = as + p,\ m = lab + pb + qa, 0 \le p < a,\ 0 \le q < b$ and

$$(2) \iff \begin{cases} t + s = l,\ t, s \ge 0 \\ at + p \le k(bs + q) \end{cases} \iff \begin{cases} s = l - t \\ 0 \le t \le \min\left\{ l, \left\lfloor \dfrac{kbl + kq - p}{kb + a} \right\rfloor \right\} \end{cases}.$$

Since, system (2) is solvable if and only if $0 \le kbl + kq - p$ then (2) is unsolvable if and only if $m = lab + pb + qa,\ 0 \le p < a, 0 \le q < b$ and

$$kbl + kq - p \le -1 \iff lb + q \le \left\lfloor \frac{p-1}{k} \right\rfloor \iff lab + qa \le a \left\lfloor \frac{p-1}{k} \right\rfloor.$$

Then m attains its greatest value if $p = a - 1$. Thus, the maximum value of m is

$$a \left\lfloor \frac{a-2}{k} = \right\rfloor + b(a-1).$$

Problem 4 Find $\mu(a_1, a_2, \ldots, a_n)$ if $a_i = a + (i-1)b, i = 1, 2, \ldots, n$ where $a > 1$ and $a \perp b$, i.e., find greatest value of m for which the equation

$$ax_1 + (a+b)x_2 + \cdots + (a+(n-1)b)x_n = m \tag{3}$$

has no solutions in $\mathbb{N}_0$.

Solution. Let $y = x_1 + x_2 + \cdots + x_n$ and $x = x_2 + 2x_3 + \cdots + (n-1)x_n, t_1 = x_n, t_2 = x_{n-1} + x_n, \ldots,$ $t_{n-2} = x_3 + \ldots + x_n, t_{n-1} = x_2 + x_3 + \ldots + x_n$. We can rewrite (3) as $ay + bx = m$, where x and y are subject to the conditions $x = t_1 + t_2 + \cdots + t_{n-1}$ and $0 \le t_1 \le t_2 \le \cdots \le t_{n-1} \le y$. Since system

$$\begin{cases} t_1 + t_2 + \cdots + t_{n-1} = x \\ 0 \le t_1 \le t_2 \le \cdots \le t_{n-1} \le y \end{cases}$$

is solvable if and only if $0 \le x \le (n-1)y$ (prove it!) then $\mu(a_1, a_2, \ldots, a_n)$ equals the greatest value of m for which the system

$$\begin{cases} ay + bx = m \\ 0 \le x \le (n-1)y \end{cases} \tag{4}$$

is unsolvable. Thus, by Problem 3,

$$\mu(a, a+b, a+2b, \ldots, (a+(n-1)b) = a\left\lfloor \frac{a-2}{n-1} \right\rfloor + b(a-1).$$

Note that since (4) solvable if and only if $(n-1)bl + (n-1)q - p \ge 0$ and $0 \le p \le a-1$, $0 \le q \le b-1$ then

$$\begin{aligned} &G(a, a+b, a+2b, \ldots, (a+(n-1)b) \\ &= \left\{ abl + pb + qa \mid 0 \le p \le a-1, 0 \le q \le b-1 \text{ and } l \ge \left\lfloor \frac{p + (b-q)(n-1) - 1}{b(n-1)} \right\rfloor \right\}. \end{aligned}$$

Problem 5 (IMO 24,1983, problem 3) Let a, b, c be pairwise coprime positive integers. Prove that

$$2abc - ab - bc - ca$$

is the largest integer which cannot be expressed as $xbc + yca + zab$, where x, y, z are nonnegative integers (or using our terminology, prove that $\mu(ab, bc, ca) = 2abc - ab - bc - ca$).

Solution. We will prove a generalization of the original problem; more precisely, we will find $\mu(A_1, A_2, \ldots, A_n)$, where $A_i = \dfrac{A}{a_i}, i = 1, 2, \ldots, n,\ A = a_1 a_2 \cdots a_n$, and $\{a_1, a_2, \ldots, a_n\}$ is a totally coprime set of positive integers, that is we will find largest value of integer m for which the equation

$$\sum_{i=1}^{n} A_i x_i = m \iff \sum_{i=1}^{n} \frac{x_i}{a_i} = \frac{m}{A} \tag{5}$$

has no solution in nonnegative integers. By the **GR** and the **GRT** there are unique integers $t_i \ge 0, p_i, r_i, i = 1, 2, \ldots, n$ such that $x_i = t_i a_i + p_i, 0 \le p < a_i, i = 1, 2, \ldots, n$ and $\dfrac{m}{A} = k + \sum_{i=1}^{n} \dfrac{r_i}{a_i}$, where $0 \le r_i < a_i, i = 1, 2, \ldots, n$. Then by uniqueness in GRT we have

$$\begin{aligned} (5) &\iff \sum_{i=1}^{n}\left(t_i + \frac{p_i}{a_i}\right) = k + \sum_{i=1}^{n} \frac{r_i}{a_i} \iff \sum_{i=1}^{n} t_i + \sum_{i=1}^{n} \frac{p_i}{a_i} = k + \sum_{i=1}^{n} \frac{r_i}{a_i} \\ &\iff \begin{cases} \sum_{i=1}^{n} t_i = k \\ p_i = r_i, i = 1, 2, \ldots, n \end{cases}. \end{aligned}$$

Since the equation $\sum_{i=1}^{n} t_i = k$ is solvable in nonnegative integers if and only if $k \geq 0$ then (5) is unsolvable if and only if

$$m \in \left\{ A\left(k + \sum_{i=1}^{n} \frac{r_i}{a_i}\right) \mid k \leq -1 \text{ and } 0 \leq r_i < a_i \right\}.$$

Hence,

$$\mu(A_1, A_2, \ldots, A_n) = A\left(-1 + \sum_{i=1}^{n} \frac{a_i - 1}{a_i}\right) = (n-1)A - \sum_{i=1}^{n} A_i.$$

Remark . Since the equation $\sum_{i=1}^{n} t_i = k$ has exactly $\binom{k+n-1}{k}$ nonnegative integer solutions, then the equation (5) with $m = Ak + \sum_{i=1}^{n} \left(\frac{r_i}{a_i}\right)$ has exactly $\binom{k+n-1}{k}$ nonnegative integer solutions as well. Prove it!

Two associated problems.

(a) Find all m for which $15x + 10y + 6z = m$ has 2010 nonnegative integer solutions.

(b) Find least m for which the equation $15x + 10y + 6z = m$ have 171 nonnegative integer solutions.

Problem 6 Let a, b be coprime positive integers. Find largest integer value m, for which the equation

$$a^2x + aby + b^2z = m \tag{6}$$

has no solutions in nonnegative integers, i.e., find $\mu(a^2, ab, b^2)$.

Solution. By the **RT** there are unique $t \geq 0, s \geq 0, p, q$ such that $x = bt + q'$, $z = as + p'$ and $0 \leq p' < a$, $0 \leq q' < b$. Then

$$a^2x + aby + b^2z = a^2(bt + q') + aby + b^2(as + p') = ab\left((at + y + bs) + \frac{bp'}{a} + \frac{aq'}{b}\right).$$

On the other hand by the **WGR** with weights (a, b) there are unique integers k, p, q such that

$$\frac{m}{ab} = k + \frac{bp}{a} + \frac{aq}{b},$$

where $0 \leq p < a, 0 \leq q < b$. Due to the uniqueness in the **WGR** we have

$$a^2x + aby + b^2z = m \iff (at + y + bs) + \frac{bp'}{a} + \frac{aq'}{b} = k + \frac{bp}{a} + \frac{aq}{b} \iff at + y + bs = k$$

and $p' = p, q' = q$. Since the equation $at + y + bs = k$ is solvable in $\mathbb{N}_0$ if and only if $k \geq 0$ then (6) is unsolvable if and only if $k(m) \leq -1$, and therefore, the largest integer value of m, for which the equation (6) has no solutions in $\mathbb{N}_0$ is

$$ab\left(-1 + \frac{b(a-1)}{a} + \frac{a(b-1)}{b}\right) = -ab + b^2(a-1) + a^2(b-1) = ab(a+b-1) - a^2 - b^2.$$

Thus,

$$\mu(a^2, ab, b^2) = ab(a+b-1) - a^2 - b^2.$$

For the interested reader the following works present other approaches to solving the proposed problems.

Further Reading

[1] J.L. Ramirez Alfonson, The Diophantine Frobenius Problem, Oxford University Press.

[2] Darren C. Ong, Vadim Ponomarenko. The Frobenius number of geometric sequences., Electronic Journal of Combinatorical Number Theory 8 (2008), Nr. A33.

[3] Amitabha Tripathi, On the Frobenius number for geometric sequences., Electronic Journal of Combinatorical Number Theory 8 (2008), Nr. A43.

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